

UNCLASSIFIED

AD 409 174

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

AD No. **409174**
DDC FILE COPY

409174

⑤ 234 300

N-63-4-2



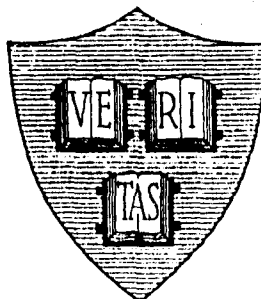
Office of Naval Research

Contract Nonr-1866(16)

NR-372-012

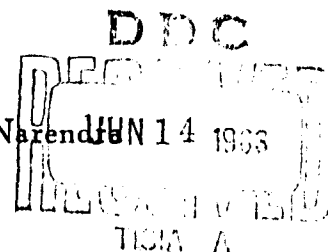
STABILITY OF CERTAIN NONLINEAR
DIFFERENTIAL EQUATIONS USING THE
SECOND METHOD OF LIAPUNOV

scale 2



By

Roger M. Goldwyn and Kumpati S. Narendra



March 18, 1963

Technical Report No. 403

Cruft Laboratory
Harvard University
Cambridge, Massachusetts

\$4.60

④ 4.60

⑤ 234 300 7-9 NA

Office of Naval Research

Contract Nonr-1866(16)

NR - 372 - 012

Technical Report

on

⑥
STABILITY OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS
USING THE SECOND METHOD OF LIAPUNOV,

by

⑩

Roger M. Goldwyn and Kumpati S. Narendra.

⑪

March 18, 1963,

⑫ 40p. ⑬ NA

The research reported in this document was made possible through support extended Cruft Laboratory, Harvard University, jointly by the Navy Department (Office of Naval Research), the Signal Corps of the U. S. Army, and the U. S. Air Force, under ONR Contract Nonr-1866(16). Reproduction in whole or in part is permitted for any purpose of the United States Government.

⑮

⑭

Technical Report No. 403

⑯ Proj. NR-372-01-

17-17 NA

Cruft Laboratory

Harvard University

Cambridge, Massachusetts

2011

21 NA

Ydc

ABSTRACT

In this report an approach is proposed for the construction of Liapunov functions for certain types of second and third-order nonlinear systems. If the system is described by a vector differential equation, $\dot{x} = F(k_1, k_2, \dots, k_m, x)$, where k_1, k_2, \dots, k_m are parameters of the system, a Liapunov function $V = x^T P x$ (P is a constant matrix) which ensures stability of the linear system for all values of k_1, k_2, \dots, k_m is called a common Liapunov function (CLF) in the given range. While it may prove difficult to determine such a CLF, a Liapunov function $V = x^T P(k_1, k_2, \dots, k_m) x$ may be selected to ensure the stability of the linear system over the entire range of the parameters. Under certain conditions, this Liapunov function may be easily modified for use as a Liapunov function for a nonlinear system in which the k_i 's are functions of the state variables.

Using this approach, sufficient conditions are determined for the stability of a differential equation of the form

$$\ddot{x} + f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x})\dot{x} + h(x)x = 0$$

in terms of the bounds on certain functions derived from f , g and h .

STABILITY OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS
USING THE SECOND METHOD OF LIAPUNOV

by

Roger M. Goldwyn and Kumpati S. Narendra
Division of Engineering and Applied Physics
Harvard University, Cambridge, Massachusetts

I. INTRODUCTION

In recent years considerable interest has been shown in the study of the stability of dynamical systems using the second method of Liapunov. Using this approach, the stability of a dynamical system is assured by the determination of a positive definite function $V(\underline{x})$ whose derivative with respect to time is negative definite. The conventional approach is one of cut and try where a positive definite function $V(\underline{x})$ is chosen as a candidate for a Liapunov function and the parameters of the system are adjusted to make $\dot{V}(\underline{x})$ negative definite. While systematic ways of generating Liapunov functions do exist for linear time-invariant systems (yielding Routh-Hurwitz conditions implicitly), the generation of such functions for nonlinear systems is more difficult. Recently, the "variable gradient method" [Ref. 3] was introduced as a systematic way of generating Liapunov functions. While this approach may, in principle, be used for any order system, in practice, even for a third-order system, it becomes fairly involved, since no set procedure exists for the choice of the elements of the gradient to satisfy the above conditions.

In this report a method related to the concept of a common Liapunov function for a linear system is presented for obtaining Liapunov functions for nonlinear systems of low order. Since, for a linear system, it is possible to derive necessary and sufficient conditions for stability, the present approach relates the nonlinear problem to a corresponding linear problem for which stability conditions are well known. If a linear system is described by a set of differential equations $\dot{\underline{x}} = \underline{F} \underline{x}$, $V(\underline{x}) = \underline{x}' \underline{P} \underline{x}$ is a Liapunov function if \underline{Q} is positive definite where $\dot{V}(\underline{x}) = \underline{x}' [\underline{F}' \underline{P} + \underline{P} \underline{F}] \underline{x} = - \underline{x}' \underline{Q} \underline{x}$. While every positive definite \underline{P} need not yield a positive definite \underline{Q} , the solution of the equation

$$\underline{F}' \underline{P} + \underline{P} \underline{F} = - \underline{Q}$$

for any positive definite \underline{Q} must yield a positive definite \underline{P} if the system is stable. This important result is used throughout the development of the present report.

II. COMMON LIAPUNOV FUNCTIONS

Consider the linear system of equations

$$\dot{\underline{x}} = \underline{F} \underline{x} \quad (1)$$

where \underline{F} is a constant $(n \times n)$ matrix and \underline{x} an $(n \times 1)$ vector. Let \underline{F} be dependent linearly on a parameter k where k is assumed to be in the range

$$\underline{k} \leq k \leq \bar{k}$$

If $V(\underline{x}) = \underline{x}' \underline{P} \underline{x}$ where \underline{P} is a positive definite (or > 0) matrix is considered as a candidate for a Liapunov function

$$\dot{V}(\underline{x}) = - \underline{x}' \underline{Q} \underline{x} \quad (2)$$

where

$$-\underline{Q} = \underline{F}' \underline{P} + \underline{P} \underline{F} \quad (3)$$

If Eq. 1 is asymptotically stable in the given range, then for any symmetric $\underline{Q} > 0$ the unique solution of Eq. 3 for \underline{P} yields a $\underline{P} > 0$. Since $\underline{F} = \underline{F}(k)$, the resulting \underline{P} will, in general, be a function of k —i.e., $\underline{P} = \underline{P}(k)$. Eq. 3 may be considered as a mapping of the set $\{\underline{Q}\}$ of positive definite matrices into the set $\{\underline{P}(k)\}$ of positive definite matrices. If there exists a matrix \underline{P}^* which is a member of $\{\underline{P}(k)\}$ such that

$$\underline{Q}^*(k) = -[\underline{F}'(k) \underline{P}^* + \underline{P}^* \underline{F}(k)] \quad (4)$$

is positive definite for all k in the given range, then

$$V^*(\underline{x}) = \underline{x}' \underline{P}^* \underline{x} \quad (5)$$

is a Liapunov function for Eq. 1 and will be termed a common Liapunov function (CLF) in the range ($\underline{k} \leq k \leq \bar{k}$) [Ref. 1].

A. CLF for First-Order System

Consider the first-order system shown in Fig. 1.

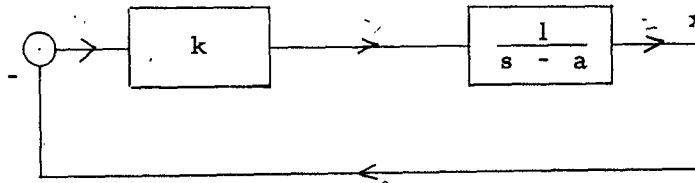


Figure 1. First-Order System

The system may be described by the equation

$$\dot{x} = ax - kx \quad (6)$$

Consider the function

$$V = px^2 \quad (7)$$

Then

$$\dot{V} = 2px [ax - kx] = 2p(a - k)x^2 = -qx^2 \quad (8)$$

It is clear that Eq. 6 is asymptotically stable for $k > a$ or, for any $q > 0$, p of Eq. 7 is > 0 in the region $k \geq \underline{k} > a$. In order to obtain a CLF in the given range, p must be chosen independent of k .

Choosing

$$p = (\underline{k} - a) \quad (9)$$

$V = (\underline{k} - a)x^2$ is a CLF for the system in the range $\underline{k} > a$ since

$$\dot{V} = 2(\underline{k} - a)(a - k)x^2 = -qx^2 \quad q > 0 \quad (10)$$

If $k = k(x)$, $V = (\underline{k} - a)x^2$ may still be considered as a CLF if $\underline{k} = \min_x [k(x)]$, and $\underline{k} > a$ since

$$\dot{V} = 2(\underline{k} - a)(a - k(x))x^2 < 0 \quad (11)$$

The nonlinear equation is asymptotically stable in the large for $\underline{k} > a$.

This simple example indicates how, for certain nonlinearities, a CLF for a linear system may also be used as a Liapunov function for the nonlinear system.

B. CLF for Second-Order System

The simple example of the preceding section was presented to indicate the first step in an approach that might hopefully yield a logical procedure for the derivation of results for a third-order system. The difficulties of such an approach become obvious even when considering a simple second-order system (Fig. 2).

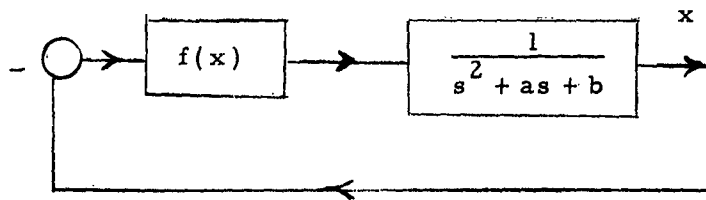


Figure 2. A Second-Order System

The second-order system shown in Fig. 2 may be represented by the set of equations

$$\dot{x}_1 = x_2 \quad (12)$$

$$\dot{x}_2 = -k(x_1)x_1 - ax_2$$

where $k(x_1) = b + f(x_1)$. In terms of Eq. 1

$$\underline{F} = \begin{bmatrix} 0 & 1 \\ -k(x_1) & -a \end{bmatrix} \quad (13)$$

When k is treated as a parameter, the range of interest is $k \geq 0$. [For $k = 0$ one of the eigenvalues is zero so that the system is stable, but not asymptotically stable.]

If $V = \underline{x}' \underline{P} \underline{x}$, then $\dot{V} = -\underline{x}' \underline{Q} \underline{x}$ where

$$\underline{Q}(k) = \begin{bmatrix} 2kp_{12} & kp_{22} - p_{11} + ap_{12} \\ kp_{22} - p_{11} + ap_{12} & 2ap_{22} - 2p_{12} \end{bmatrix} \quad (14)$$

If a common Liapunov function V exists, it must be possible to select \underline{P}

from $\underline{Q}(0)$, since $k=0$ lies in the allowable range of k .

$$\underline{Q}(0) = \begin{bmatrix} 0 & ap_{12} - p_{11} \\ ap_{12} - p_{11} & 2(ap_{22} - p_{12}) \end{bmatrix} \quad (15)$$

$\underline{Q}(0)$ can only be positive semi-definite and, hence, one must select

$$q_{12} = q_{21} = 0 \text{ or}$$

$$ap_{12} = p_{11} \quad (16)$$

Hence,

$$\underline{Q}(k) = \begin{bmatrix} 2kp_{12} & kp_{22} \\ kp_{22} & 2(ap_{22} - p_{12}) \end{bmatrix} \quad (17)$$

For positive definiteness of $\underline{Q}(k)$ it is necessary that

$$(i) \ p_{12}, \ ap_{22} - p_{12} > 0 \quad (18)$$

$$(ii) \ \det \underline{Q}(k) = q(k) > 0.$$

$$q(k) = -k^2 p_{22}^2 + 4kp_{12}(ap_{22} - p_{12}) \quad (19)$$

It should be observed that it is impossible to get a CLF of the assumed quadratic type for all $k > 0$, since for some value of k , $q(k)$ must become negative. The maximum value of k for which a CLF may be determined is given by

$$\bar{k} = \frac{4p_{12}(ap_{22} - p_{12})}{p_{22}^2} \quad (20)$$

Any matrix \underline{P} which satisfies conditions Eqs. 16 and 18 is positive definite for $k, a > 0$. In particular, it is of interest to select a \underline{P} which is independent of k , since $\underline{x}'\underline{P}\underline{x}$ is then a Liapunov function of the nonlinear system.

Since \underline{P} is to be a constant matrix, selecting arbitrarily $p_{22} = 2$ and maximizing Eq. 20, $p_{12} = a$ and

$$V = \underline{x}' \begin{bmatrix} a^2 & a \\ a & 2 \end{bmatrix} \underline{x} \quad (21)$$

is a CLF for the range $0 \leq k \leq \bar{k} = a^2$ and indicates stability of the nonlinear system when $k = k(x_1)$.

C. The Liapunov Function $V = \underline{x}'\underline{P}(k)\underline{x}$

The inability to obtain a CLF of the assumed quadratic form $(\underline{x}'\underline{P}\underline{x})$ for the linear problem in the entire range $k(x_1) > 0$ is a serious drawback to the method. The difficulty may arise due to the restrictive nature of the definition of a CLF, i. e., the assumption that $V(x)$ is a quadratic form. The off-diagonal term kp_{22} of Eq. 17 is responsible for the upper bound on k . Additional freedom may be obtained by permitting p_{11} to be linear in k , i. e.,

$$p_{11} = c + vk \quad (22)$$

where c and v are constants. Thus, $\underline{Q}(k)$ of Eq. 14 may be expressed as

$$\underline{Q}(k) = \begin{bmatrix} 2kp_{12} & kp_{22} - c - vk + ap_{12} \\ kp_{22} - c - vk + ap_{12} & 2ap_{22} - 2p_{12} \end{bmatrix} . \quad (23)$$

Since $k = 0$ lies in the allowable range, it is necessary that

$$c = ap_{12} . \quad (24)$$

With Eq. 24,

$$q(k) = \det \underline{Q}(k) \quad (25)$$

becomes

$$q(k) = - (kp_{22} - vk)^2 + 4kp_{12} (ap_{22} - p_{12}) . \quad (26)$$

If one selects

$$v = p_{22} , \quad (27)$$

then $q(k) > 0$ implies no upper bound on k as Eq. 20 did, since

$$q(k) = 4kp_{12} (ap_{22} - p_{12}) . \quad (28)$$

Choosing $p_{22} = 2$ and $p_{12} = a$

$$q(k) = 4ka^2 . \quad (29)$$

The elements of the \underline{P} matrix are

$$p_{11} = a^2 + 2k , \quad p_{12} = a , \quad p_{22} = 2 . \quad (30)$$

The procedure followed so far is to select $\underline{Q}(k)$ in such a manner as to be least restrictive on the parameter k . Since $\underline{Q}(k)$ is positive definite for all $k > 0$, $\underline{P}(k)$ must also be positive definite for all $k > 0$, since it is merely the solution of Eq. 3 for this specific example.

$$V(\underline{x}) = \underline{x}' \begin{bmatrix} a^2 & a \\ a & 2 \end{bmatrix} \underline{x} + 2kx_1^2 \quad (31)$$

$$= \underline{x}' \underline{P}_0 \underline{x} + 2kx_1^2 \quad (32)$$

where \underline{P}_0 is a constant matrix independent of k . For $k = k(x_1)$ the second term in $V(\underline{x})$ may be replaced by

$$4 \int_0^{x_1} uk(u) du \quad (33)$$

so that,

$$V(\underline{x}) = \underline{x}' \underline{P}_0 \underline{x} + 4 \int_0^{x_1} uk(u) du \quad (34)$$

and

$$\dot{V}(\underline{x}) = -\underline{x}' \underline{Q}(k) \underline{x}$$

where

$$\underline{Q}(k) = \begin{bmatrix} 2ak & 0 \\ 0 & 2a \end{bmatrix} \quad (35)$$

It is important to realize what the modification of Eq. 34 has accomplished and under what conditions such a modification is possible. Since the linear system is asymptotically stable for $k > 0$, any $\underline{Q}(k) > 0$ must yield a $\underline{P}(k) > 0$. In the second-order system considered above, $\underline{P}(k)$ could be written as \underline{P}_0 —which is independent of k —together with a matrix which has

all elements zero except the element in the first row and first column which is $2k$. In replacing Eq. 32 by 34 it is necessary that k be a function of x_1 only since the time derivative of Eqs. 32 and 34 yield identical \underline{Q} 's considered as functions of k . Further, for $k \geq 0$ Eqs. 32 and 34 also obey

$$V(\underline{x}) \geq \underline{x}' \underline{P}_0 \underline{x} \quad (36)$$

\underline{P}_0 is a positive-definite matrix since it is a solution of Eq. 15 for $k = 0$. Hence, $V(\underline{x})$ of Eq. 34 is a positive-definite function.

D. Generalization

The results obtained in the preceding section may be generalized to the case of a system with several parameters k_1, k_2, \dots, k_m . For the linear system one can attempt to find a Liapunov function of the form

$$V(\underline{x}) = \underline{x}' \underline{P}_0 \underline{x} + \sum_{i=1}^m v_i k_i x_i^2 \quad (37)$$

where \underline{P}_0 is independent of the k_i 's and v_i are constants ($i=1 \dots m, m \leq n$). For the nonlinear system, if $k_i = k_i(x_i)$, the Liapunov function may be modified to

$$V(\underline{x}) = \underline{x}' \underline{P}_0 \underline{x} + 2 \sum_{i=1}^m v_i \int_0^{x_i} k_i(u) u \, du \quad (38)$$

The total time derivative of Eqs. 37 and 38 yield identical \underline{Q} 's as functions of the k_i 's.

Further,

$$V(\underline{x}) \geq \underline{x}' \underline{P}_0 \underline{x} + \sum_{i=1}^m v_i k_i^* x_i^2 \quad (39)$$

where

$$k_i^* = \begin{cases} \text{Max}_{x_i} [k_i(x_i)] & v_i < 0 \\ \text{Min}_{x_i} [k_i(x_i)] & v_i > 0 \end{cases} \quad (40)$$

Eq. 39 is clearly greater than zero, since it is a special case of Eq. 37 which is greater than zero ($\|\underline{x}\| \neq 0$).

It is also possible to make a further generalization using LaSalle's extension [2] of Liapunov's theorem. According to LaSalle's theorem, it is permissible for $\dot{V} \leq 0$ as long as \dot{V} does not vanish identically on a trajectory. This implies that \underline{Q} may be chosen to be only positive semi-definite. For the second-order system of Eq. 23, if

$$\underline{Q}(k) = \begin{bmatrix} 0 & 0 \\ 0 & 4a \end{bmatrix}, \quad (41)$$

then,

$$p_{11} = c + vk, \quad p_{12} = 0, \quad 2a p_{22} = 4a \quad (42)$$

and, hence,

$$c = 0, \quad v = p_{22}, \quad p_{22} = 2. \quad (43)$$

For the nonlinear case we have the Liapunov function

$$V(\underline{x}) = 4 \int_0^{x_1} u k(u) du + 2x_2^2 \quad (44)$$

and

$$\dot{V}(\underline{x}) = -4ax_2^2. \quad (45)$$

Hence, even for the case when $\underline{Q}(k)$ is only positive semi-definite the remarks of the preceding sections apply.

E. Summary

The results of this section may be summarized as follows: Consider the differential equation $\dot{\underline{x}} = \underline{F} \underline{x}$ which depends linearly on parameters k_1, k_2, \dots, k_m . If the solutions of the equations are stable for $\underline{k}_i \leq k_i \leq \bar{k}_i$

1. $V = \underline{x}' \underline{P}(k_1, \dots, k_m) \underline{x}$ is a Liapunov function if \underline{P} is the solution of the equation

$$\underline{F}'(k_1, \dots, k_m) \underline{P} + \underline{P} \underline{F}(k_1, \dots, k_m) = - \underline{Q}(k_1, \dots, k_m) \quad (46)$$

2. If a $\underline{Q} > 0$ can be selected such that \underline{P} is independent of the k_i 's, $V(\underline{x}) = \underline{x}' \underline{P} \underline{x}$ is a common Liapunov function in the range $\underline{k}_i \leq k_i \leq \bar{k}_i$ and, consequently, for the nonlinear system where $k_i = k_i(\underline{x})$. However, in most cases such a Liapunov function can rarely be obtained.

3. A modification (which may be used for the nonlinear system) is possible if $k_i = k_i(x_i)$. In such a case if a $\underline{Q} > 0$ is selected so that the solution of Eq. 46 yields a $\underline{P}(k_1, \dots, k_m)$ such that p_{ij} is a constant for $i \neq j$ and p_{ii} is a linear function of k_i , the Liapunov function for the nonlinear system is as shown in Eq. 38. It is greater than zero for $\|\underline{x}\| \neq 0$ as it may be bounded as in Eq. 39 by a quadratic form coming from the solution of Eq. 46 for a particular value of the constants k_i in the allowable range.

F. Conclusion

The procedure presented in this section is seen to be similar to the "variable gradient method" [3] developed recently. In the conventional approach a $V(\underline{x}) > 0$ is selected as a candidate for a Liapunov function and the conditions the parameters of the system have to satisfy to make $\dot{V}(\underline{x}) < 0$ are determined to establish the stability conditions. In the "variable gradient method" the gradient ∇V of V is chosen to make $\dot{V} = \nabla V' \dot{\underline{x}}$ negative definite. $\frac{n(n-1)}{2}$ curl conditions have to be satisfied by ∇V to make it the gradient of some potential function $V(\underline{x})$. One must then insure that $V(\underline{x}) > 0$ for $\|\underline{x}\| \neq 0$.

In the approach presented in this section, the nonlinear problem is related to the corresponding linear problem (when all the parameters are constant) for which the stability conditions are known. By selecting a Liapunov function for the linear problem and using the procedure indicated previously, one is assured that $V(\underline{x})$ is positive and $\dot{V}(\underline{x})$ negative definite even for the nonlinear system.

In the following sections, the above approach is applied to several problems. Modifications to the procedure are also suggested - always insuring that the conditions for the stability of the nonlinear system reduce to the Routh-Hurwitz conditions when the parameters are constant.

III. THE SECOND-ORDER SYSTEM

In this section, conditions are obtained for the stability of certain second-order differential equations. The first two results have been discussed by other authors [2] but are included here for convenience. The conditions obtained for the problem in section C are, hopefully, new and are less restrictive than those known to the authors.

$$A. \ddot{x} + f(x, \dot{x})\dot{x} + g(x)x = 0 \quad (47)$$

The usual state vector representation for Eq. 47 with $x = x_1$ is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -gx_1 - fx_2 \end{aligned} \quad (48)$$

For this case

$$\underline{Q}(f, g) = \begin{bmatrix} 2gp_{12} & gp_{22} - p_{11} + fp_{12} \\ gp_{22} - p_{11} + fp_{12} & 2fp_{22} - 2p_{12} \end{bmatrix} \quad (49)$$

For f and g constant the Routh-Hurwitz conditions demand that $f, g \geq 0$ for stability or $f, g > 0$ for asymptotic stability. Hence, $f = 0$ may be assumed to be in the allowable range.

$$\underline{Q}(0, g) = \begin{bmatrix} 2gp_{12} & gp_{22} - p_{11} \\ gp_{22} - p_{11} & -2p_{12} \end{bmatrix} \quad (50)$$

Since for $g > 0$ $\underline{Q}(0, g)$ can never be positive definite, one must take

$\underline{Q}(0, g) \equiv 0$ so that

$$p_{12} = 0 \quad (51)$$

$$p_{11} = p_{22} g \quad (52)$$

Since $g = g(x_1)$ and p_{22} is assumed to be a constant condition, Eq. 52 can be satisfied. With Eqs. 51 and 52, Eq. 50 reduces to

$$\underline{Q}(f, g) = \begin{bmatrix} 0 & 0 \\ 0 & 2fp_{22} \end{bmatrix} \quad (53)$$

so that one may take $q_{22} = 2f$ and, hence, $p_{22} = 1$. Thus, for Eq. 48, one has

$$V = 2 \int_0^{x_1} u g(u) du + x_2^2 \quad (54)$$

and

$$\dot{V} = -2f(x_1, x_2) x_2^2 \quad (55)$$

\dot{V} may not vanish identically on a trajectory for $f > 0$ and $g(x_1) \neq 0$ for $x_1 \neq 0$. Thus, Eq. 54 indicates asymptotic stability in the large for

$$(i) f(x_1, x_2) > 0$$

$$(ii) g(x_1) \neq 0 \text{ for } x_1 \neq 0 \quad (56)$$

$$(iii) \int_0^{x_1} u g(u) du \rightarrow \infty \text{ as } |x_1| \rightarrow \infty$$

(iii) of Eq. 56 with (ii) demands that $g(x_1) > 0$ for $x_1 \neq 0$. [The argument starting with $\underline{Q}(f, 0)$ was essentially given in the previous section].

$$B. \ddot{x} + f(x)\dot{x} + g(x)x = 0 \quad (57)$$

This is a special case of Eq. 47 but has been included here to indicate the importance of the choice of state variables while determining a Liapunov function. In a practical situation the "average" dissipation may be more significant than the instantaneous dissipation and the Liénard transformation [4] may be used. Defining

$$\int_0^{x_1} f(u) du = F(x_1) \quad (58)$$

Eq. 57 reduces to

$$\begin{aligned} \dot{x}_1 &= -\hat{f}(x_1)x_1 + x_2 \\ \dot{x}_2 &= -gx_1 \end{aligned} \quad (59)$$

where

$$\hat{f}(x_1) = \frac{F(x_1)}{x_1} \quad (60)$$

For this case,

$$\underline{Q}(\hat{f}, g) = \begin{bmatrix} 2(\hat{f}p_{11} + gp_{12}) & \hat{f}p_{12} + gp_{22} - p_{11} \\ \hat{f}p_{12} + gp_{22} - p_{11} & -2p_{12} \end{bmatrix} \quad (61)$$

Selecting

$$\underline{Q}(\hat{f}, g) = \begin{bmatrix} q_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad (62)$$

which is positive semi-definite (or ≥ 0) for $q_{11} > 0$, it is found that $p_{12} = 0$ and $p_{11} = gp_{22}$. These conditions are compatible for $p_{22} = \text{constant}$. Then, with $q_{11} = 2\hat{f}g$, one obtains $p_{22} = 1$. Thus,

$$V = 2 \int_0^{x_1} u g(u) du + x_2^2 \quad (63)$$

and

$$\dot{V} = -2g\hat{f}x_1^2 = -2gF(x_1)x_1 \quad (64)$$

Equation 63 indicates asymptotic stability in the large for

$$\begin{aligned} \text{(i)} \quad \hat{f}(x_1) &= \frac{F(x_1)}{x_1} > 0 & \text{for } x_1 \neq 0 \\ \text{(ii)} \quad g(x_1) &> 0 & \text{for } x_1 \neq 0 \end{aligned} \quad (65)$$

$$\text{(iii)} \quad \int_0^{x_1} u g(u) du \rightarrow \infty \quad \text{as } |x_1| \rightarrow \infty$$

With Eq. 65, \dot{V} will not vanish identically except on the $\underline{x} \equiv 0$ trajectory.

Furthermore, Eq. 65 does not demand that the instantaneous dissipation be > 0 , but only that the "average" dissipation be > 0 . As an example of this, consider

$$g(x_1) = 1$$

$$f(x_1) = (x_1 - 2)(x_1 + 2)(x_1 - \sqrt{3})(x_1 + \sqrt{3}) = x_1^4 - 7x_1^2 + 12$$

Then

$$\hat{f}(x_1) = \frac{x_1^4}{5} - \frac{7x_1^2}{3} + 12$$

which is > 0 for real x_1 . Notice $f(x_1)$ is negative for $-2 < x_1 < -\sqrt{3}$ and $\sqrt{3} < x_1 < 2$.

C. Second-Order Nonlinear Feedback System

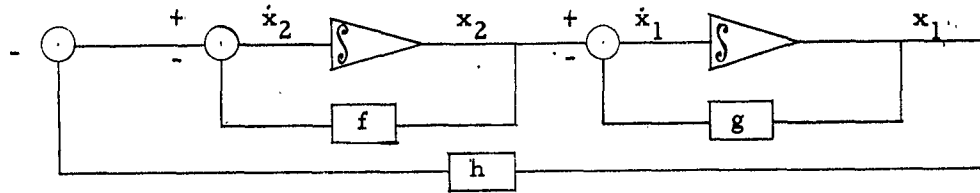


Figure 3. Second-Order Nonlinear Feedback System

The system of Fig. 3 may be described by the set of equations

$$\begin{aligned}\dot{x}_1 &= -g(x_1)x_1 + x_2 \\ \dot{x}_2 &= -h(x_1)x_1 - f(x_2)x_2\end{aligned}\quad (66)$$

This may be reduced to the single second-order differential equation

$$\ddot{x} + \left[g(x) + g'(x)x + \left(1 + \frac{g(x)x}{\dot{x}} \right) f(\dot{x} + g(x)x) \right] \dot{x} + h(x)x = 0 \quad (67)$$

where $x = x_1$.

Eq. 67 is the standard form of Eq. 47. There are two immediate objections to applying the results of A. A perhaps minor objection is that Eq. 67 contains a derivative of the function g . It would be desirable not to demand that the nonlinearities be differentiable functions. A more serious objection and one in fact that prohibits the application of A, is that the "dissipation" of Eq. 67 is not of constant sign in the vicinity of $\dot{x} = 0$.

Examining Eq. 67 for the case of f, g, h constant, one obtains

$$\ddot{x} + (f + g) \dot{x} + (fg + h) x = 0 \quad . \quad (68)$$

The system is thus stable for

$$\begin{aligned} (i) \quad fg + h &> 0 \\ (ii) \quad f + g &> 0 \end{aligned} \quad . \quad (69)$$

As has been previously mentioned, one wishes to select the $\underline{Q}(f, g, h)$ for the nonlinear system which will be no more restrictive than Eq. 69 when applied to a linear system.

$$\underline{Q}(f, g, h) = \begin{bmatrix} 2(p_{11}g + p_{12}h) & p_{22}h + p_{12}(f+g) - p_{11} \\ p_{22}h + p_{12}(f+g) - p_{11} & 2(p_{22}f - p_{12}) \end{bmatrix} \quad . \quad (70)$$

As in Eq. 23, the off-diagonal elements of Eq. 70 will introduce undesirable restrictions on f, g , and h . For f, g, h constant, one could select

$$\underline{Q}(f, g, h) = \begin{bmatrix} q_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad (71)$$

and, hence, obtain

$$\begin{aligned} p_{12} &= p_{22} f \\ p_{11} &= p_{22} h + p_{12}(f+g) \\ q_{11} &= 2(p_{11}g + p_{12}h) \end{aligned} \quad . \quad (72)$$

If

$$q_{11} = 4(f+g)(fg+h) \quad (73)$$

then the conditions of Eq. 69 are clearly displayed. With Eqs. 72 and 73, one finds

$$\begin{aligned} p_{11} &= 2h + 2f(f + g) \\ p_{12} &= 2f \\ p_{22} &= 2 \end{aligned} \quad (74)$$

For the nonlinear problem, however, since $f = f(x_2)$, one is unable to select p_{11} and p_{12} as in Eq. 74. Realizing that Eq. 69 implies that for $g > 0$ any f greater than some minimum is satisfactory, one may try

$$\begin{aligned} p_{11} &= 2h + 2\underline{f}(\underline{f} + g) \\ p_{12} &= 2\underline{f} \\ p_{22} &= 2 \end{aligned} \quad (75)$$

where

$$\underline{f} = \min_{x_2} [f(x_2)] \quad (76)$$

Notice that even for f, g, h constants and $g < 0$ and $h > 0$, f from Eq. 69 is not permitted to increase without bound.

With Eq. 75, 70 becomes

$$\underline{Q} = \begin{bmatrix} 2(h + \underline{f}g)(\underline{f} + g) & \underline{f}(f - \underline{f}) \\ \underline{f}(f - \underline{f}) & 2(f - \underline{f}) \end{bmatrix} \quad (77)$$

which is at least ≥ 0 for

$$(h + \underline{f}g)(\underline{f} + g) > \frac{f^2(f - \underline{f})}{4} \geq 0 \quad (78)$$

since $f \geq \underline{f}$. If one defines

$$\bar{f} = \text{Max}_{x_2} [f(x_2)] \quad (79)$$

then the system is asymptotically stable in the large for

$$(i) \quad \underline{f} g(x_1) + h(x_1) > 0$$

$$(ii) \quad [f + g(x_1)] [f g(x_1) + h(x_1)] > \frac{f^2(\bar{f} - \underline{f})}{4} \geq 0 \quad (80)$$

and

$$(iii) \quad \int_0^{x_1} [\underline{f} g(u) + h(u)] u du \rightarrow \infty \text{ as } |x_1| \rightarrow \infty$$

and

$$V = 2 \int_0^{x_1} [\underline{f} g(u) + h(u)] u du + (\underline{f} x_1 + x_2)^2 \quad (81)$$

Eq. 80 reduces to the conditions of Eq. 69 for f constant even with $g = g(x_1)$ and $h = h(x_1)$ and also permits a negative $g(x_1)$. (ii) of Eq. 80 may be used to determine the maximum value $f(x_2)$ may assume while still being guaranteed stability by Eq. 81. It is clear that the conditions of Eq. 69 are necessary to allow for the validity of Eq. 80. Further, notice that it was not necessary to introduce the derivatives of any of the functions through a Jacobian matrix [5]. This was partially due to the form of \underline{F} , i.e., $F_{ij} = F_{ij}(x_j)$ only.

As an example of the above, take

$$f(x_1) = 2 + a e^{-|x_1|}, \quad a \geq 0$$

$$g(x_1) = -e^{-x_1^2}$$

and

$$h(x_1) = 6 + 2|x_1|$$

Then $\underline{f} = 2$ and $\bar{f} = 2 + a$ for $a \geq 0$. It is desired to find how large a may be taken. Now $(\underline{f}g + h) \geq (-2 + 6) = 4 > 0$ and $\underline{f} + g \geq 2 - 1 = 1 > 0$. Thus,

$$(\underline{f}g + h)(\underline{f} + g) \geq 4.$$

One may select, therefore, from Eq. 80

$$\frac{\underline{f}^2(\bar{f} - \underline{f})}{4} = 3,$$

and, hence,

$$\bar{f} = \underline{f} + 3 = 5,$$

so that stability is assured for $0 \leq a \leq 3$.

IV. THIRD-ORDER SYSTEM

In this section, conditions for the stability of the differential equation

$$\ddot{x} + f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x})\dot{x} + h(x)x = 0 \quad (82)$$

are considered. This equation can be represented in the standard state vector form as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 \quad (83)$$

$$\dot{x}_3 = -h(x_1)x_1 - g(x_1, x_2)x_2 - f(x_1, x_2, x_3)x_3$$

where $x = x_1$.

When f , g and h are constants, the Routh-Hurwitz conditions for stability are

$$\begin{aligned} \text{(i)} \quad & f, g, h > 0 \\ \text{(ii)} \quad & fg - h > 0 \end{aligned} \quad (84)$$

The conditions for stability of the nonlinear problems to be derived below are such that they reduce to Eq. 84 when f , g and h are constant.

$$A. \quad \ddot{x} + f(\dot{x})\dot{x} + g(\dot{x})\dot{x} + x = 0$$

Using the above notation,

$$\underline{Q}(f, g, h) = \begin{bmatrix} 2p_{13}h & p_{13}g + p_{23}h - p_{11} & p_{33}h + p_{13}f - p_{12} \\ p_{13}g + p_{23}h - p_{11} & 2(p_{23}g - p_{12}) & p_{33}g + p_{23}f - p_{22} - p_{13} \\ p_{33}h + p_{13}f - p_{12} & p_{33}g + p_{23}f - p_{22} - p_{13} & 2(p_{33}f - p_{23}) \end{bmatrix} \quad (85)$$

The problem at this stage is to determine \underline{Q} so that it contains in a readily recognizable form the conditions of Eq. 84 if f , g , h were constants.

One may thus proceed as in Section III-C and attempt to employ a diagonal

\underline{Q} . Noticing that $q_{23} = q_{32}$ contains $f(x_2)$, $g(x_2)$, and p_{22} , it is possible to make these terms zero. Thus, one tries

$$\underline{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix} \quad (86)$$

Solving Eq. 85 with 86 ,

$$\begin{aligned}
 p_{11} &= p_{23} h \\
 p_{12} &= p_{33} h \\
 p_{13} &= 0 \\
 p_{22} &= p_{33} g + p_{23} f \\
 q_{22} &= 2 (p_{23} g - p_{33} h) \\
 q_{33} &= 2 (p_{33} f - p_{23} h)
 \end{aligned} \tag{87}$$

Selecting

$$q_{22} = 2 (mg - h) \tag{88}$$

and

$$q_{33} = 2 (f - m)$$

where m is a constant, then

$$\begin{aligned}
 p_{23} &= m \\
 p_{33} &= 1
 \end{aligned} \tag{89}$$

Since \underline{Q} must be positive definite, it is necessary that

$$(i) \underline{f} - m \geq 0$$

and

$$(ii) m \underline{g} - h \geq 0$$

where

$$\underline{f} = \min_{x_2} [f(x_2)] \tag{91}$$

and

$$\underline{g} = \min_{x_2} [g(x_2)] \tag{92}$$

[Equality is not permitted for both (i) and (ii) of Eq. 90] . m may be selected to maximize

$$q = (\underline{f} - m) (m\underline{g} - h) \quad (93)$$

or

$$m = \frac{\underline{f}\underline{g} + h}{2\underline{g}} \quad (94)$$

With Eq. 94 , Eq. 90 reduces to

$$\begin{aligned} \text{(i)} \quad \frac{\underline{f}\underline{g} - h}{2\underline{g}} &\geq 0 \\ \text{(ii)} \quad \frac{\underline{f}\underline{g} - h}{2} &\geq 0 \end{aligned} \quad (95)$$

Thus, for $\underline{Q} \geq 0$, it is necessary that

$$\begin{aligned} \text{(i)} \quad \underline{g} &> 0 \\ \text{(ii)} \quad \underline{f}\underline{g} - h &> 0 \end{aligned} \quad (96)$$

For (ii) of Eq. 96 to hold for $h = 0$, one must also have $\underline{f} > 0$. Taking $h > 0$, it is assured that $m > 0$ which is required by $p_{11} = mh$.

In summary, the system is asymptotically stable in the large for

$$\begin{aligned} \text{(i)} \quad \underline{f}, \underline{g}, h &> 0 \\ \text{(ii)} \quad \underline{f}\underline{g} - h &> 0 \end{aligned} \quad (97)$$

where

$$\begin{aligned} V = m h x_1^2 + 2 h x_1 x_2 + 2 \int_0^{x_2} [g(u) + m f(u)] u du \\ + 2 m x_2 x_3 + x_3^2 \end{aligned} \quad (98)$$

and m is given by Eq. 94. Further,

$$V \geq \underline{x}' \underline{P}^* \underline{x} \quad (99)$$

where

$$\underline{P}^* = \begin{bmatrix} mh & h & 0 \\ h & (\underline{g} + m\underline{f}) & m \\ 0 & m & 1 \end{bmatrix} \quad (100)$$

which is positive definite. With Eq. 97, \dot{V} may not vanish identically except on the $\underline{x} \equiv 0$ trajectory. For f and g constant, conditions Eq. 97 are identical to those of Eq. 84.

$$B. \ddot{x} + f(\dot{x})\dot{x} + g(\dot{x})\ddot{x} + h(x)x = 0$$

This case is to be identical to that of A, except here h is not a constant but $h = h(x_1)$. If Eq. 98 were independent of h , then it would also be a Liapunov function for this case. However, Eq. 98 is a function of h but fortunately only in what corresponds to the $p_{11}x_1^2$ term and the $2p_{12}x_1x_2$ term (taking m as constant). A modification of these terms may be attempted in a manner that will not drastically alter the \underline{Q} of A.

By examining Eqs. 85 and 87, it is clear that the proper generalization of the $p_{11}x_1^2$ term is to replace it as before by

$$2 \int_0^{x_1} p_{11}(u) u \, du$$

since it came from $p_{11} = p_{23}h$. For this case then, one might try this as the only change taking

$$V = 2m \int_0^{x_1} h(u) u \, du + 2h(x_1) x_1 x_2 + 2 \int_0^{x_2} [g(u) + mf(u)] u \, du + 2mx_2 x_3 + x_3^2 \quad (101)$$

It is clear that m has to be redefined from Eq. 94 for it is required to be constant.

Trying Eq. 101, one obtains

$$\dot{V} = -\underline{x}' \underline{Q} \underline{x} \quad (102)$$

where

$$\underline{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(mg - H') & 0 \\ 0 & 0 & 2(f - m) \end{bmatrix} \quad (103)$$

$$H(x_1) = h(x_1) x_1, \quad (104)$$

and

$$H' = \frac{dH}{dx_1} \quad (105)$$

Thus, for $\underline{Q} \geq 0$ even in the most restrictive case, it is necessary that

$$\begin{aligned} \text{(i)} \quad \underline{f} - m &\geq 0 \\ \text{and} \\ \text{(ii)} \quad m\underline{g} - \overline{H'} &\geq 0 \end{aligned} \quad (106)$$

where

$$\overline{H'} = \text{Max}_{x_1} [H'(x_1)] \quad (107)$$

Here again equality is not permitted for both (i) and (ii) of Eq. 106. If m is selected as in A to maximize the product of (i) and (ii) of Eq. 106, then

$$m = \frac{\underline{f}\underline{g} + \overline{H'}}{2\underline{g}} \quad (108)$$

With Eqs. 106 and 108, following A, for $\underline{Q} \geq 0$, it is necessary that

$$\begin{aligned} \text{(i)} \quad \underline{g} &> 0 \\ \text{(ii)} \quad \underline{f}\underline{g} - \overline{H'} &> 0 \end{aligned} \quad (109)$$

If (ii) of Eq. 109 is to hold for $h \equiv 0$, one must have $\underline{f} > 0$. Here $\overline{H'} > 0$ guarantees an $m > 0$.

With m defined by Eq. 106 and taking V of Eq. 101 as the Liapunov function,

$$V \geq 2m \int_0^{x_1} H(u) du + 2H(x_1)x_2 + \underline{g}x_2^2 + m\underline{f}x_2^2 + 2mx_2x_3 + x_3^2 \quad (110)$$

or

$$V \geq \left[\frac{H(x_1)}{\sqrt{\underline{g}}} + \sqrt{\underline{g}}x_2 \right]^2 + [x_2, x_3] \begin{bmatrix} m\underline{f} & m \\ m & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \frac{2}{\underline{g}} \int_0^{x_1} [m\underline{g} - H'(u)]h(u)u du \quad (111)$$

With Eq. 109, $m > 0$, and $h(x_1) > 0$ for $x_1 \neq 0$, V of Eq. 111 is > 0 for $\|\underline{x}\| \neq 0$.

To summarize, the system is asymptotically stable in the large for

$$\begin{aligned} \text{(i)} \quad h(x_1) &> 0 \text{ for } x_1 \neq 0 \\ \text{(ii)} \quad \underline{f}, \underline{g}, \overline{H'} &> 0 \\ \text{(iii)} \quad \underline{f}\underline{g} - \overline{H'} &> 0 \\ \text{(iv)} \quad \int_0^{x_1} H(u) du &\rightarrow \infty \text{ as } |x_1| \rightarrow \infty \end{aligned} \quad (112)$$

(iv) of Eq. 112 insures that $V \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$. Also notice that \dot{V} may only vanish identically on the $\underline{x} \equiv 0$ trajectory.

C. The General Case $\ddot{x} + f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x})\dot{x} + h(x)x = 0$.

It is desirable that the results of this section include as a special case the results of the sections A and B. This implies that a Liapunov function must be obtained which would reduce identically to that of Eq. 101 for $f \equiv f(x_2)$ and $g \equiv g(x_2)$. To this end, one anticipates a redefinition of m of Eq. 108 and a change in the integral involving f and g . Thus try

$$V = 2m \int_0^{x_1} H(u) du + 2H(x_1)x_2 + 2 \int_0^{x_2} [\hat{g}(u) + m\hat{f}(u)] u du + 2mx_2x_3 + x_3^2 \quad (113)$$

where $\hat{g}(x_2)$ and $\hat{f}(x_2)$ are to be determined in a manner to reduce identically to g and f when they are functions of x_2 only.

Now

$$\dot{V} = -\underline{x}' \underline{Q} \underline{x} \quad (114)$$

where

$$\underline{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(mg - H') & (g - \hat{g}) + m(f - \hat{f}) \\ 0 & (g - \hat{g}) + m(f - \hat{f}) & 2(f - m) \end{bmatrix} \quad (115)$$

To insure that $\underline{Q} \geq 0$, one demands that

$$(i) \quad mg - \bar{H}' > 0$$

$$(ii) \quad \underline{f} - m > 0$$

and

$$(iii) \quad q = 4(mg - \bar{H}')(\underline{f} - m) - \text{Max} [(g - \hat{g}) + m(f - \hat{f})]^2 > 0 \quad (116)$$

where here the $\underline{\quad}$ means the absolute minimum of the particular function, and $\overline{H'}$ was defined in Eq. 107. If Eq. 116 is satisfied, then $\underline{Q} \geq 0$.

(iii) of Eq. 116 is the requirement that was not present in the previous sections. This condition will be the least restrictive if one selects \hat{g} and \hat{f} so as to minimize $\text{Max} [(g - \hat{g}) + m(f - \hat{f})]^2$. With the stipulation on \hat{g} and \hat{f} of the first paragraph of C, it is natural to make the following definitions:

$$\begin{aligned}\overline{f}(x_2) &= \text{Max}_{x_1, x_3} [f(x_1, x_2, x_3)] \\ \underline{f}(x_2) &= \text{Min}_{x_1, x_3} [f(x_1, x_2, x_3)] \\ \hat{f}(x_2) &= \frac{\overline{f}(x_2) + \underline{f}(x_2)}{2}\end{aligned}\tag{117}$$

or

$$\begin{aligned}\widetilde{f} &= \text{Max}_{x_2} \left[\text{Max}_{x_1, x_3} |f(x_1, x_2, x_3) - \hat{f}(x_2)| \right] \\ \widetilde{f} &= \text{Max}_{x_2} \left[\frac{\overline{f}(x_2) - \underline{f}(x_2)}{2} \right] \\ \underline{\underline{f}} &= \text{Min}_{x_2} [f(x_2)]\end{aligned}$$

The same notation is adopted for $g(x_1, x_2)$, e. g. $\overline{g}(x_2) = \text{Max}_{x_1} [g(x_1, x_2)]$, to define $\overline{g}(x_2)$, $\underline{g}(x_2)$, $\hat{g}(x_2)$, \widetilde{g} , and $\underline{\underline{g}}$. $\overline{H'}$ has been defined previously.

With the above, (iii) of Eq. 116 requires that

$$q = 4(m\underline{\underline{g}} - \overline{H'}) (\underline{\underline{f}} - m) - (\widetilde{g} + m\widetilde{f})^2 > 0 \tag{118}$$

If now m is selected to maximize q of Eq. 118, it is found that

$$m = \frac{2\underline{f}\underline{g} + 2\overline{H}^I - \widetilde{f}\widetilde{g}}{\widetilde{f}^2 + 4\underline{g}} \quad (119)$$

With m taken as in Eq. 119, it is found that $q > 0$ requires

$$\underline{f}\underline{g} - \overline{H}^I > \epsilon \quad (120)$$

where

$$\epsilon = \sqrt{\widetilde{f}^2 \underline{f} \overline{H}^I + \widetilde{f} \widetilde{g} (\underline{f}\underline{g} + \overline{H}^I) + \widetilde{g}^2 \underline{g}} \quad (121)$$

To summarize, the following conditions insure that Eq. 82 is asymptotically stable in the large:

- (i) $h(x_1) > 0$ for $x_1 \neq 0$
 - (ii) $\underline{f}, \underline{g}, \overline{H}^I > 0$
 - (iii) $\underline{f}\underline{g} - \overline{H}^I > \epsilon \geq 0$
 - (iv) $\int_0^{x_1} H(u) du \rightarrow \infty$ as $|x_1| \rightarrow \infty$
- (122)

where ϵ is defined in Eq. 120 and the Liapunov function is defined in Eq. 113 with m given by Eq. 119. With Eq. 122, one can show as in B that $V > 0$ for $||\underline{x}|| \neq 0$ since Eq. 122 implies $m > 0$. Further, \dot{V} may only vanish identically on the $\underline{x} \equiv 0$ trajectory.

The conditions for stability of Eq. 122 and ϵ in Eq. 121 may appear complicated. The following four cases, however, indicate that these

conditions may be reduced to simpler forms for specific types of nonlinearities and do contain the results of A and B.

1. Take f , g , and h as constants. Then $\tilde{f} \equiv \tilde{g} \equiv \epsilon \equiv 0$, and Eq. 122 reduces to the Routh-Hurwitz conditions of Eq. 84.
2. Take f constant, $g = g(\dot{x})$, and $h = h(x)$. Then $\tilde{f} \equiv \tilde{g} \equiv \epsilon \equiv 0$, and Eq. 122 becomes Barbashin's result [6].
3. Take $f = f(\dot{x})$, $g = g(\dot{x})$, and $h = h(x)$. Then $\epsilon \equiv 0$ and Eq. 122 is just Eq. 112 of B. For $h = \text{constant}$, Eq. 122 becomes Eq. 97 of A.
4. Cases 2 and 3 considered above require conditions similar to those of a linear system. If $g = g(\dot{x})$, $f = f(x, \dot{x})$, and $h = h(x)$ an ϵ may be determined satisfying (iii) of Eq. 122. Since, in this case, $\tilde{g} = 0$ and $\underline{f} = \underline{f}$,

$$2\tilde{f} = \bar{f} - \underline{f} = \frac{2\epsilon}{\sqrt{\underline{f}H}}$$

and the condition reduces to

$$\underline{f} \leq f(x, \dot{x}) \leq \bar{f} = \underline{f} + \frac{2\epsilon}{\sqrt{\underline{f}H}}$$

[Numerical example: assume $\underline{f} = 1$, $\underline{g} = 7$, and $H = 4$ so that $\underline{f}g = H = 3 > 0$. Taking $\epsilon = 2.99$ it is seen that $1 \leq f(x, \dot{x}) \leq 3.99$ insures stability.]

Consider the following example which indicates an application of the most general results. Assume

$$f(x_1, x_2, x_3) = ae^{-[x_1^2 + x_2^2 + x_3^2]} + 1 + \frac{a}{|x_2| + 1}$$

$$g(x_1, x_2) = 6 + be^{-[x_1^6 + x_2^2]}$$

$$+ \frac{2b}{x_2^4 + 2}$$

$$h(x_1) = 1 - \frac{3}{x_1} [1 - (1 + x_1) e^{-x_1}]$$

$$\text{for } x_1 > 0 \text{ and } h(-x_1) = h(x_1)$$

It is desired to determine the range of a and b so that stability is assured.

Using the previous definitions

$$\bar{f}(x_2) = ae^{-x_2^2} + 1 + \frac{a}{|x_2| + 1}$$

$$\underline{f}(x_2) = 1$$

$$\hat{f}(x_2) = \frac{a}{2} \left[e^{-x_2^2} + \frac{1}{|x_2| + 1} \right] + 1$$

$$\tilde{f} = \text{Max}_{x_2} \frac{a}{2} \left[e^{-x_2^2} + \frac{1}{|x_2| + 1} \right] = a$$

$$\underline{\underline{f}} = 1$$

$$\bar{g}(x_2) = 6 + be^{-x_2^2} + \frac{2b}{x_2^4 + 2}$$

$$\begin{aligned} \underline{g}(x_2) &= 6 \\ \hat{g}(x_2) &= 6 + \frac{b}{2} \left[e^{-x_2^2} + \frac{2}{x_2^4 + 2} \right] \\ \tilde{g} &= \max_{x_2} \frac{b}{2} \left[e^{-x_2^2} + \frac{2}{x_2^4 + 2} \right] = b \\ \underline{g} &= 6 \end{aligned}$$

Also

$$H(x_1) = x_1 - 3[1 - (1 + x_1)e^{-x_1}] \quad \text{for } x_1 > 0$$

and

$$\begin{aligned} H(-x_1) &= -H(x_1) \\ H'(x_1) &= 1 - 3x_1 e^{-x_1} \quad \text{for } x_1 > 0 \end{aligned}$$

and

$$\begin{aligned} H'(-x_1) &= H'(x_1) \\ H'' &= 1 \end{aligned}$$

Thus, $\underline{f}\underline{g} - H'' = 5 > 0$ and one may select $\epsilon < 5$. Then with Eq. 121,

$$a^2 + 7ab + 6b^2 < 25$$

The region of allowable a and b is shown in Fig. 4.

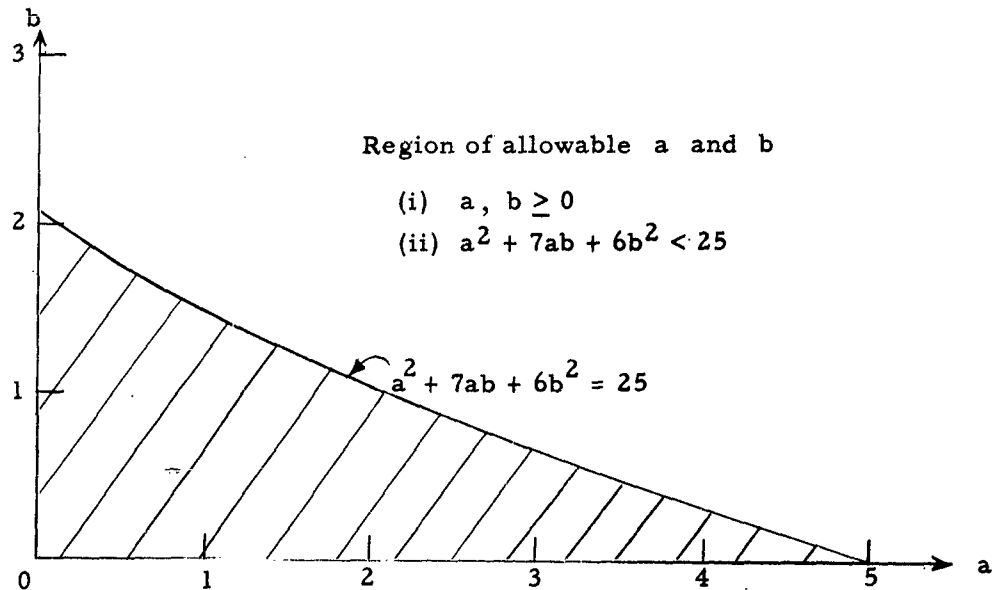


Figure 4. Region of Permissible Values of a and b

For example, one may take

1. $a = 0$ and $b < \frac{5}{\sqrt{6}}$
2. $b = 0$ and $a < 5$
3. $a = b < \frac{5}{\sqrt{14}}$

All the conditions of Eq. 122 have been obeyed. Notice that $H'(x_1)$ is negative for certain values x_1 , e. g., $|x_1| = 1$.

$$D. \ddot{x} + f\dot{x} + g(x)\dot{x} + hx = 0 \quad (123)$$

In this section, f and g are constants. Of course, Eq. 123 may be considered as a special case of C . However, less stringent conditions will be derived for stability than those given by Eq. 122 for this special case.

The following state vector form is taken which may be considered as a logical extension to Eq. 123 of the Liénard transformation of Section III - B for a second-order system:

$$\begin{aligned} \dot{y}_1 &= y_3 \\ \dot{y}_2 &= -hy_1 \\ \dot{y}_3 &= -G(y_1) + y_2 - fy_3 \end{aligned} \quad (124)$$

where

$$y_1 = x$$

and

$$G(y_1) = \int_0^{y_1} g(u) du \quad (125)$$

Proceeding as in A,

$$\underline{\underline{Q}} \left(\frac{G(y_1)}{y_1} \right) = \begin{bmatrix} 2 \left[hp_{12} + \frac{G}{y_1} p_{13} \right] & hp_{22} + \frac{G}{y_1} p_{23} - p_{13} & hp_{23} + \frac{G}{y_1} p_{33} + fp_{13} - p_{11} \\ hp_{22} + \frac{G}{y_1} p_{23} - p_{13} & -2p_{23} & fp_{23} - p_{12} - p_{33} \\ hp_{23} + \frac{G}{y_1} p_{33} + fp_{13} - p_{11} & fp_{23} - p_{12} - p_{33} & 2(fp_{33} - p_{13}) \end{bmatrix} \quad (126)$$

Notice that for g constant, $\frac{G(y_1)}{y_1} = g$.

One may select

$$\underline{Q} = \begin{bmatrix} 2(f \frac{G}{y_1} - h) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (127)$$

so that

$$\begin{aligned} p_{11} &= f^2 + \frac{G}{y_1} & p_{12} &= -1 \\ p_{22} &= \frac{f}{h} & p_{13} &= f \\ p_{33} &= 1 & p_{23} &= 0 \end{aligned} \quad (128)$$

and, hence,

$$V = f^2 y_1^2 + 2 \int_0^{y_1} G(u) du - 2y_1 y_2 + \frac{f}{h} y_2^2 + 2f y_1 y_3 + y_3^2 \quad (129)$$

or

$$V = \frac{2}{f} \int_0^{y_1} [f \frac{G(u)}{u} - h] u du + \frac{f}{h} (y_2 - \frac{h}{f} y_1)^2 + (f y_1 + y_3)^2. \quad (130)$$

Thus, to summarize, Eq. 123 is asymptotically stable in the large for

$$\begin{aligned} \text{(i)} \quad & f, \frac{G(y_1)}{y_1}, h > 0 \quad \text{for } y_1 \neq 0 \\ \text{(ii)} \quad & f \frac{G(y_1)}{y_1} - h > 0 \\ \text{(iii)} \quad & \int_0^{y_1} [f \frac{G(u)}{u} - h] u du \rightarrow \infty \quad \text{as } |y_1| \rightarrow \infty \end{aligned} \quad (131)$$

Further, $V > 0$ for $||\underline{y}|| \neq 0$ and that with (iii) of Eq. 131, $V \rightarrow \infty$ as $||\underline{y}|| \rightarrow \infty$. Also, with Eq. 131, \dot{V} may only vanish identically on the $\underline{y} \equiv 0$ trajectory.

It is required by Eq. 131 that only $\frac{G(y_1)}{y_1}$ be > 0 and not that $g(y_1) > 0$. As an example of this, consider

$$f = 36$$

$$g(y_1) = (y_1 - 2)(y_1 + 2)(y_1 - \sqrt{3})(y_1 + \sqrt{3}) = y_1^4 - 7y_1^2 + 12$$

$$h = 186$$

Then

$$\frac{G(y_1)}{y_1} = \frac{y_1^4}{5} - \frac{7y_1^2}{3} + 12$$

which is > 0 for y_1 real. Further, the minimum of $\frac{G(y_1)}{y_1}$ is $\frac{187}{36}$.

Thus, $f \frac{G(y_1)}{y_1} - h > f \frac{187}{36} - h = 1 > 0$. All the conditions of Eq. 131 are, hence, satisfied.

V. CONCLUSIONS

The various results presented throughout this report were derived by considering a CLF (or a modification of the CLF) for a related linear system. All the results for the nonlinear equations reduce to the usual Routh-Hurwitz conditions when the nonlinearities are assumed constant.

VI. REFERENCES

1. K. S. Narendra and Y. C. Ho, "On the Construction of Liapunov Functions for Nonlinear Systems, " Cruft Lab. Tech. Report No. 328, Harvard University, Cambridge, Massachusetts (September 15, 1960).
2. J. LaSalle and S. Lefschetz, Stability by Liapunov's Direct Method (With Applications), Academic Press, New York, New York, 1961.
3. D. G. Schultz and J. E. Gibson, "The Variable Gradient Method for Generating Liapunov Functions, " Applications and Industry of IEEE (September, 1962).
4. For example : Nicholas Minorsky, Nonlinear Oscillations, D. van Nostrand Co., Princeton, New Jersey, 1962, p. 104.
5. N. N. Krasovski, "On the Stability in the Large of a System of Nonlinear Differential Equations, " Prinp. Mat. Mekh., vol. 18, pp. 735-737 (1954).
6. E. A. Barbashin, "On the Stability of Solution of a Third-Order Nonlinear Differential Equation, " Prinp. Mat. Mekh., vol. 16, pp. 629-632 (1952).

TA

DISTRIBUTION LIST

Activity Supply Officer
Building 2504, Charles Wood Area
Fort Monmouth, New Jersey (50)
Attn: Director of Research

Commanding Officer
Office of Naval Research
Navy 100, Box 39 (55)
First Post Office
New York, New York

Armed Services
Technical Information Agency
Arlington Hall Station (10)
Arlington 15, Virginia
Attn: TIPDR

The Director
Naval Research Laboratory
Washington 25, D. C. (6)
Attn: Technical Information Office

Commander, AF CRL
AFRD, ARDC, CRRLC
Laurence G. Hanscom Field (4)
Bedford, Massachusetts
Attn: Electronics Research Directorate

Commanding General
Air Research and Development Command
P. O. Box 1393 (3)
Baltimore 1, Maryland
Attn: RDTRAP

Chief of Naval Research
Department of the Navy
Washington 25, D. C. (2)
Attn: Dr. A. Shostak, Code 427

Chief of Naval Research
Department of the Navy
Washington 25, D. C. (2)
Attn: Code 427

Commanding Officer
Office of Naval Research
495 Summer Street (2)
Boston, Massachusetts

Chief, Bureau of Ships
Department of the Navy (2)
Washington 25, D. C.
Attn: Code 810

Director, Air University
Library (2)
Maxwell Air Force Base
Alabama

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 421

Commanding Officer
Office of Naval Research
495 Summer Street
Boston, Massachusetts

Commanding Officer
Office of Naval Research
John G. Lewis Library Building
84 East Randolph Street
Chicago 1, Illinois

Commanding Officer
Office of Naval Research
36 Broadway
New York 13, New York

Commanding Officer
Office of Naval Research
1000 East Green Street
Pasadena, California

Commanding Officer
Office of Naval Research
1000 Geary Street
San Francisco 7, California

Head, Document Section
Technical Information Division
Naval Research Laboratory
Washington 25, D. C.

Martin A. Gorden
Magnesium Branch, Code 6150
Solid State Division
Naval Research Laboratory
Washington 25, D. C.

Commanding Officer
U. S. N. Air Development Center
Johnsville, Pennsylvania
Attn: WAGO Library

Commander
U. S. N. Air Development Center
Johnsville, Pennsylvania
Attn: A&E

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: EL-1

Engineering Librarian
Convair
San Diego 12, California

Dr. John E. Hipkin
Applied Physics and Ferrite Division
Rensselaer Polytechnic Inst.
P. O. Box 1824
Troy, New York

Engineering Librarian
Sperry Microwave Electronics Co.
Clearwater, Florida

Dr. Lajos Rinal
Research Division
Raytheon Company
Waltham 54, Massachusetts

Elizabeth Weeks, Librarian
Raytheon Company
18 Seyon Street
Waltham 54, Massachusetts

Report Librarian
Sylvania Electric Products Inc.
Electronic Systems Division
100 First Avenue
Waltham, Massachusetts

Document Control Center
Wayland Library
Raytheon Manufacturing Co.
Wayland, Massachusetts

J. E. Goldman
Scientific Laboratory
Ford Motor Company
Engineering Staff
P. O. Box 2053
Dearborn, Michigan

Charles C. H. Tang
Bell Telephone Labs.
Murray Hill, New Jersey

Librarian
RCA Laboratories
Princeton, New Jersey

Dr. A. Smith
RCA
Princeton, New Jersey

Commander (2)
U. S. Naval Electronics Lab.
San Diego, California

Commanding General, RCW
Rome Air Development Center
Griffis Air Force Base (2)
Rome, New York

Commanding General
Air Research and Development Command
P. O. Box 1395 (2)
Baltimore, Maryland
Attn: RDDDE

Commander
Air Force Cambridge Research Labs.
Bedford, Massachusetts
Attn: CROTL2

Commander
Wright Air Development Center
Wright Patterson Air Force Base
Ohio (2)
Attn: WCLRA Library

National Security Agency
Physical Sciences Division (2)
Fort George Meade, Maryland

Associate Prof. A. Kaprielian
Department of Electrical Engineering
University of Southern California
University Park
Los Angeles, California

Assistant Secretary of Defense
(Research and Development)
Research and Development Board
Department of Defense
Washington 25, D. C.

Chief of Naval Operations
Department of the Navy
Washington 25, D. C.
Attn: Op-10

Chief of Naval Operations
Department of the Navy
Washington 25, D. C.
Attn: Op-11

Chief of Naval Operations
Department of the Navy
Washington 25, D. C.
Attn: Op-41

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: EL-4

Technical Library
U. S. Naval Proving Ground
Dahlgren, Virginia

Director
Naval Ordnance Laboratory
White Oak, Maryland

Librarian
U. S. Naval Post Graduate School
Monterey, California

Air Force Office of Scientific Research
Air Research and Development Command
Washington 25, D. C.
Attn: RFL, Physics Division

Commanding General
Rome Air Development Center
Griffis Air Force Base
Rome, New York
Attn: RCRCO-4C

Commanding General
Rome Air Development Center
Griffis Air Force Base
Rome, New York
Attn: RCR

Commander
Air Force Cambridge Research Center
210 Albany Street
Cambridge 39, Massachusetts
Attn: CR21

Commander
Air Force Cambridge Research Center
310 Albany Street
Cambridge 39, Massachusetts
Attn: CR21M

Commander
AF Cambridge Research Laboratories
Laurence G. Hanscom Field
Bedford, Massachusetts
Attn: Dr. Hollingsworth

Commander
Wright Air Development Center
Wright Patterson Air Force Base
Ohio
Attn: WCR

Sandia Corporation
Org. 1446, Sandia Base
Albuquerque, New Mexico
Attn: Dr. W. Harrison, Jr.

Sandia Corporation
Sandia Base
Albuquerque, New Mexico
Attn: Library Division 1922-1

Mr. Robert Turner
General Electric Company
Advanced Electronics Center
Ithaca, New York

Library
Airborne Instruments Lab.
Walt Whitman Road
Melville, Long Island, New York

Secretary, Working Group
Semiconductor Devices
346 Broadway, 8th Floor
New York 19, New York
Attn: ACCT

Metals Research Laboratories
Electro Metallurgical Company
Box 580, Niagara Falls, New York
Attn: Mr. R. J. Gladieux

Librarian
General Electric Research Lab.
Schenectady, New York
P. O. Box 1068

Westinghouse Electric Corp.
Research Laboratories
Beulah Road, Churchill Boro.
Pittsburgh 35, Pennsylvania

Prof. O. E. H. Rydbeck
P. O. Box 246
Belmar, New Jersey

Dr. John W. Aaron
611 East Genesee Street
Fayetteville, New York

Librarian
Airborne Instruments
Minneapolis, Minnesota

Commander
Wright Air Development Center
Wright Patterson Air Force Base
Ohio
Attn: WCRH

Commander
AF Institute of Technology
Wright Patterson Air Force Base
Ohio

Headquarters
AF Institute of Technology
MU-155, ARDC
Patterson Air Force Base
Florida

U. S. Coast Guard
1300 E Street, N. W.
Washington 25, D. C.
Attn: EEE

M. A. Krivonozh, Chief
Systems Component Branch
Electronic Warfare Division
Signal Corps Agency
White Sands Proving Ground
New Mexico

Mr. A. D. Dedrosian
Signal Corps Liaison Office
Maine Institute of Technology
Building 28, Room 131
Cambridge 39, Massachusetts

Chief, European Office
ARDC Command
Shell Building
60 Rue Ravenstein
Brussels, Belgium

Dr. J. Anton Hoffman
Ordnance Materials Res. Office
Waterdown Arsenal
Waterdown, Massachusetts

Acquisitions Officer
ARTIA Reference Center
Arlington Hall Station
Arlington 15, Virginia

Stanford Research Institute
Documents Center
Menlo Park, California
Attn: Mary Lou Fields

Dr. C. H. Papp
Dept. of Electrical Engineering
Stanford Institute of Technology
Palo Alto, California

Stanford Electronics Lab.
Stanford University
Stanford, California

Applied Electronics Lab.
Stanford University
Stanford, California

Department of Electrical Engineering
Tale University
New Haven, Connecticut

Librarian
Johns Hopkins University
1315 St. Paul Street
Baltimore 2, Maryland

Radiation Laboratory
Johns Hopkins University
1315 St. Paul Street
Baltimore 2, Maryland

Director, Lincoln Laboratory
Mass. Institute of Technology
Bedford, Massachusetts

Mr. John Hewitt
Department Room
Research Lab. of Electronics
Mass. Institute of Technology
Cambridge 39, Massachusetts

Professor A. Von Hippel
Mass. Institute of Technology
Lab. for Insulation Research
Cambridge 39, Massachusetts

Library, Room A 227
Lincoln Laboratory
P. O. Box 73
Lexington 73, Massachusetts

K. J. Siegel, Head
Theory and Analysis Department
Willow Run Laboratories
University of Michigan
Willow Run Airport
Ypsilanti, Michigan

Librarian
National Bureau of Standards Library
Room 301, Northwest Building
Washington 25, D. C.

Librarian
U. S. Department of Commerce
National Bureau of Standards
Boulder, Colorado

Dr. Karl Callen
National Security Agency
Physical Sciences Division
Fort George Meade, Maryland

Dr. H. Campaigne
National Security Agency
Physical Sciences Division
Fort George Meade, Maryland

Chang Kuang University
Electrical Engineering Department
Tainan, Taiwan
Republic of China
Attn: Professor Chao-Hai Chou
Head, Eng. Department

Mr. D. S. Jones
Department of Mathematics
Univ. College of No. Staffordshire
Keele, Staffordshire, England

Professor Paul Sami Mito
Osaka City University
Dept. of Engineering Sciences
1-1 Nishi-Ogino, Kita-ku
Osaka, Japan

Donald C. Hinson
Dept. of Electrical Eng.
University of Arizona
Tucson 23, Arizona

Professor Jerome R. Singer
Div. of Electrical Engineering
University of California
Berkeley 4, California

Professor Charles Kittel
Department of Physics
University of California
Berkeley 4, California

Series Librarian
Brandeis University
Waltham, Massachusetts

Professor H. G. Buehr
School of Electrical Engineering
City University
Hoboken, New York

Library, College of Engineering
University of Illinois
Urbana, Illinois

E. A. Chapman, Librarian
Rensselaer Polytechnic Institute
Ames Hall
Troy, New York

Robert Plonsey
Department of Engineering
Case Institute of Technology
University Circle
Cleveland 6, Ohio

Dept. of Electrical Engineering
Case Institute of Technology
University Circle
Cleveland 6, Ohio

Attn: S. Seely, Head

Battelle Memorial Institute
Columbus, Ohio
Attn: Electrical Engineering Division

Librarian
Engineering Library
Brown University
Providence, Rhode Island

Professor A. W. Swette
Dept. of Electrical Engineering
University of Texas
Austin 14, Texas

Mr. William Way
Research Librarian
Texaco Instruments Corp.
3377 Culberson Boulevard
Hollywood 28, California

SOLID STATE ONLY

ELECTROMAGNETIC RADIATION ONLY

Michigan State College
Department of Mathematics
East Lansing, Michigan

Microwave Research Institute
Polytechnic Institute of Brooklyn
15 Johnson Street
Brooklyn, New York

Mr. L. J. Swartz, Jr.
Building 12, Room 2017
Hughes Research Laboratories
Culver City, California

Professor O. E. H. Rydbeck
P. O. Box 246
Belmar, New Jersey

One Copy Unless Otherwise Specified